AD-A054 918   AEROSPACE MEDICAL RESEARCH LAB WRIGHT-PATTERSON AFB OHIO   F/G 12/1
THE MOST EFFICIENT METHOD TO NUMERICALLY COMPUTE THE SCALAR SOL--ETC(U)
1976   D W REPPERGER
UNCLASSIFIED   AMRL-TR-76-88   NL

1 OF 1
AD
A054918

END
DATE
FILMED
7-78
DDC

AD A054918

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: (14) AMRL-TR-76-88	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	(2)
4. TITLE (and Subtitle): (6) THE MOST EFFICIENT METHOD TO NUMERICALLY COMPUTE THE SOLUTION OF THE STEADY STATE RICCATI Equation. Scalar	5. TYPE OF REPORT & PERIOD COVERED
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): (10) Daniel W. Repperger	8. CONTRACT OR GRANT NUMBER(s): (17) 09
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Aerospace Medical Research Laboratory, Aerospace Medical Division, Air Force Systems Command, Wright-Patterson Air Force Base, Ohio 45433	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: (16) 62202F-7222-722209-72220908
11. CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE: (11) 1976
	13. NUMBER OF PAGES: 7 (12) 8p.
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

DDC
RECEIVED
JUN 7 1978
E

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Approved for public release; distribution unlimited

18. SUPPLEMENTARY NOTES

A paper prepared for presentation and publication in Applications and Research Informantion Systems and Sciences, Volume 1, Proceedings of the First International Conference on Information Sciences and Systems, Patras, Greece, 19-24 August 1976.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Steady state
Riccati equation
Algorithm

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(See reverse side)

009850 Gu

DD FORM 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Abstract

This paper will consider the numerical calculation of the steady state riccati equation utilizing a continued fraction approach. The method considered here has many similarities with the partitioned algorithm approach [1,2] considered by D.G. Lainiotis for the solution of riccati type equations. First the matrix riccati equation in steady state is converted (by a coordinate transformation) into another riccati equation which can be expanded in a matrix continued fraction form. The transformed matrix riccati equation may not be invertible and therefore restrictions are made on the general matrix case to guarantee the proper inversion properities.

The scalar riccati equation is then considered in an effort to give incite into the solution of the matrix equation. A numerical procedure is outlined to iterate on the solution of the scalar steady state equation. Upper and lower bound approximations are obtained for the steady state case based on N terms of the continued fraction expansion. Theorems are proved on the convergence properties of the continued fraction expansion utilizing mathematical induction. By using the continued fraction algorithm, the difference (or error) between the riccati solution and the first N terms of the continued fraction expansion can be bounded from above and below.

A theorem from Number Theory demonstrates that the riccati equation is a quadratic surd and it must exhibit periodicity in its continued fraction expansion. This result was demonstrated in [1] using the partitioned algorithm method. The type of periodicity obtained here is slightly different but has many of the same similarities as obtained in [1].

If the problem of interest were to approximate the numerical solution of the riccati equation using a mathematical approach called, "Approximation by Convergents", an investigation is made on this approach of continued fractions in the calculation of an irrational number (such as π). By considering the continued fraction expansion of π using the largest reciprocals of the difference between π and its respective convergences as the elements of the fraction expansion, this method is most efficient. For the case of the riccati equation, the periodic property of the continued fraction coefficients gives convergents which are identical to the periodic terms (hence, most efficient). It is hoped that these results can be extended to the matrix case to obtain the most efficient expansion.

THE MOST EFFICIENT METHOD TO NUMERICALLY COMPUTE THE SCALAR SOLUTION OF THE STEADY STATE RICCATI EQUATION

DANIEL W. REPPERGER
Wright Patterson Air Force Base
Dayton, Ohio (U.S.A.)

Abstract

This paper will consider the numerical calculation of the steady state riccati equation utilizing a continued fraction approach. The method considered here has many similarities with the partitioned algorithm approach [1,2] considered by D.G. Lainiotis for the solution of riccati type equations. First the matrix riccati equation in steady state is converted (by a coordinate transformation) into another riccati equation which can be expanded in a matrix continued fraction form. The transformed matrix riccati equation may not be invertible and therefore restrictions are made on the general matrix case to guarantee the proper inversion properities.

The scalar riccati equation is then considered in an effort to give incite into the solution of the matrix equation. A numerical procedure is outlined to iterate on the solution of the scalar steady state equation. Upper and lower bound approximations are obtained for the steady state case based on N terms of the continued fraction expansion. Theorems are proved on the convergence properties of the continued fraction expansion utilizing mathematical induction. By using the continued fraction algorithm, the difference (or error) between the riccati solution and the first N terms of the continued fraction expansion can be bounded from above and below.

A theorem from Number Theory demonstrates that the riccati equation is a quadratic surd and it must exhibit periodicity in its continued fraction expansion. This result was demonstrated in [1] using the partitioned algorithm method. The type of periodicity obtained here is slightly different but has many of the same similarities as obtained in [1].

If the problem of interest were to approximate the numerical solution of the riccati equation using a mathematical approach called, "Approximation by Convergents", an investigation is made on this approach of continued fractions in the calculation of an irrational number (such as π). By considering the continued fraction expansion of π using the largest reciprocals of the difference between π and its respective convergences as the elements of the fraction expansion, this method is most efficient. For the case of the riccati equation, the periodic property of the continued fraction coefficients gives convergents which are identical to the periodic terms (hence, most efficient). It is hoped that these results can be extended to the matrix case to obtain the most efficient expansion.

The research reported in this paper was sponsored by Aerospace Medical Research Laboratory, Aerospace Medical Division, Air Force Systems Command, Wright Patterson Air Force Base, Ohio 45433. Further reproduction is authorized to satisfy needs of the U.S. Government.

1. Introduction

The riccati equation has been extensively studied. The approach used here will consider a continued fraction method for numerical calculation of the steady state matrix riccati equation. The continued fraction approach has similarities to the partitioned algorithm method [1,2] . By using the continued fraction approach, the main advantage occurs due to the fact that the riccati equation can be approximated by a finite number of terms of the known matrices. If a truncation is made after N terms of the continued fraction expansion, then an upper and lower bound for the difference between the actual riccati solution and the approximation can be obtained for the scalar case. The difference between the two bounds can also be calculated which gives an idea of the ability of this approach to squeeze the true riccati solution between the respective bounds. Some examples are worked to illustrate the advantages of this approach. In order to apply this approach, it is necessary for the development of basic theorems before the continued fraction method can be utilized.

2. Conversion of The Riccati Equation Into The Continued Fraction Form

The problem of interest is the solution of the following matrix equation:

$$A P + P A^T - P R P + Q = 0 \qquad (1)$$

The unknown matrix P is nxn and the matrices A,R, and Q are all nxn and known. It is initially assumed that R is positive definite; the remaining conditions

(such as P and Q both symmetric and positive definite, A canonical and stable, and the controllability and observability conditions) will not be stated initially in an effort to derive theorems based on as few assumptions as possible. Theorem 1 illustrates the method to convert the matrix riccati equation (1) into the form in which the continued fraction approach can be used.

Theorem 1: The solution to equation (1) with R positive definite can be specified as the solution of the following matrix equation:

$$W W - K W - L = 0 \qquad (2)$$

where

$$P = W R^{-1} + R^{-1} A^T \qquad (3)$$

Proof:

Substitute equation (3) into equation (1) yields:

$$AWR^{-1} + AR^{-1}A^T + WR^{-1}A^T + R^{-1}A^TA^T$$

$$-(WR^{-1}+R^{-1}A^T)R(WR^{-1}+R^{-1}A^T) + Q = 0$$

or

$$AWR^{-1} + AR^{-1}A^T + WR^{-1}A^T + R^{-1}A^TA^T - WWR^{-1}$$

$$-WR^{-1}A^T - R^{-1}A^TRWR^{-1} - R^{-1}A^TRR^{-1}A^T + Q = 0$$

simplifying yields:

$$WW + [R^{-1}A^TR - A]W + [-AR^{-1}A^TR - QR] = 0$$

Hence, if the choice is made of the form;

$$K = -R^{-1}A^TR + A \qquad (4a)$$

$$L = QR + AR^{-1}A^TR \qquad (4b)$$

then equation (2) will follow.

It is noted that the matrix K may be singular but if R is symmetric, Q positive definite and symmetric, then the matrix L will be positive definite.

To conclude the study on the matrix riccati equation, the continued fraction approach (for the matrix case) will be investigated using equation (2) in lieu of equation (1). The next theorem demonstrates the advantage of expressing equation (1) in the form of equation (2) for matrix continued fractions.

Theorem 2

If W^{-1} exists and $K \neq 0$, then

$$W = K + \cfrac{L}{K + \cfrac{L}{K + \cdots}}$$

$$= K + L[K + L[K + L[K + ---]^{-1}]^{-1}]^{-1} \qquad (5)$$

if W^{-1} exists and K=0, then $W = [L]^{1/2}$ which is different from standard matrix square root notation.

Proof:

Equation (2) can be written

$$W = K + L W^{-1} = K + \frac{L}{W}$$

Thus equation (5) follows; if K = 0, then $W = L^{1/2}$ as indicated above.

The scalar riccati equation will now be considered in an effort to study the numerical algorithm presented here and to gain incite into the solution of the matrix case.

3. The Scalar Riccati Equation

Consider the following equation:

$$aP + Pa - P^2r + q = 0$$

with

$$r > 0, \quad q > 0, \quad \text{and} \quad a < 0$$

equations (4a-b) now become:

$$K = 0$$
$$L = qr + a^2$$

equation (2) now becomes:

$$W^2 - (qr + a^2) = 0$$

which has a solution specified by:

$$W = \sqrt{qr + a^2} \qquad (6)$$

It is easily seen that W always has one positive root because:

$$qr + a^2 > 0$$

P is specified by:

$$P = \frac{W}{r} + \frac{a}{r}$$

Note that under the standard assumptions, if $W > 0$, $q > 0$, $a < 0$, and $r > 0$, it then follows that $P > 0$ always.

The continued fraction expansion for W is trivial in this case. In order to investigate the scalar riccati equation in greater detail, convergence theorems will be proven based on Euclid's Algorithm [3] with the proper notation used in continued fraction expansions.

4. Convergence Theorems on The Continued Fraction Expression For The Riccati Equation

Consider the scalar continued fraction expansion of the form:

$$W = a_o + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \cfrac{1}{a_N + \cdots}}}} \qquad (7)$$

The continued fraction notation of equation (7) will be specified by:

$$W = \langle a_o, a_1, a_2, a_3, ---, a_N, ---\rangle$$

The following definition is essential to discuss convergence properties of continued fraction expansions.

Definition:

Let a_o, a_1, a_2, --- be an infinite sequence of integers, all positive except perhaps a_o. Define two sequences of integers $\{h_N\}$ and $\{K_N\}$ inductively as follows:

$$h_{-2} = 0, \quad h_{-1} = 1, \quad h_i = a_i h_{i-1} + h_{i-2} \qquad (8a)$$

$$K_{-2} = 1, \quad K_{-1} = 0, \quad K_i = a_i K_{i-1} + K_{i-2} \qquad (8b)$$

for $i \geq 0$

Note: $K_o=1$, $K_1=a_1K_o \geq K_o$, $K_2 > K_1$

Hence: $K_N > K_{N-1} > 1$ for $N \geq 2$

$K_1 \geq K_o = 1$

Theorem 3

For any positive real number x

$$\langle a_o, a_1, ---, a_{N-1}, x\rangle = \frac{xh_{N-1} + h_{N-2}}{xK_{N-1} + K_{N-2}}$$

Proof: if N=0, implies:

$$x = \frac{xh_{-1} + h_{-2}}{xK_{-1} + K_{-2}}$$

which is true from the substitutions (8a-b).
For N=1 the result is

$$\langle a_o, x\rangle = a_o + \frac{1}{x} = \frac{xh_o + h_{-1}}{xK_o + K_{-1}}$$

which is true by the substitutions (8a-b). Using mathematical induction it is necessary to assume that the results hold for $\langle a_o, a_1, ---, a_{N-1}, x\rangle$.

To calculate the partial sum for N terms, it is necessary to use:

$$\langle a_o, a_1, ---, a_N, x\rangle = \langle a_o, a_1, ---, a_{N-1}, a_N + \frac{1}{x}\rangle$$

$$= \frac{(a_N + 1/x)h_{N-1} + h_{N-2}}{(a_N + 1/x)K_{N-1} + K_{N-2}}$$

$$= \frac{x(a_Nh_{N-1} + h_{N-2}) + h_{N-1}}{x(a_NK_{N-1} + K_{N-2}) + K_{N-1}} = \frac{xh_N + h_{N-1}}{xK_N + K_{N-1}}$$

Thus if the results hold for N-1, then they hold for N. Since they hold for N=0 and N=1, then they hold for all N. Q.E.D.

The next theorem is useful for determining upper and lower bounds for sums of the sequences.

Definition:

Define $r_N = \langle a_o, a_1, --- a_N\rangle$ for all integers $N \geq 0$. Notice that r_N is the approximation after N terms of the continued fraction expansion.

Theorem 4

$$r_N = \frac{h_N}{K_N}$$

Proof: $\langle a_o, a_1, ---, a_{N-1}, x\rangle = \dfrac{xh_{N-1} + h_{N-2}}{xK_{N-1} + K_{N-2}}$

from theorem (3). Now replace x by a_N. This implies

$$r_N = \langle a_o, a_1, ---, a_N\rangle = \frac{a_N h_{N-1} + h_{N-2}}{a_N K_{N-1} + K_{N-2}} \qquad (9)$$

but the definitions (8a-b) imply that equation (9) can be written

$$r_N = \frac{h_N}{K_N} \qquad \text{Q. E. D.}$$

The next theorem examines convergence of the continued fraction expansion from the calculation of an upper and a lower bound.

Theorem 5

As a lower bound

r_j for j even forms an increasing sequence, i.e.

$$r_o < r_2 < r_4 < r_6 < ---$$

As an upper bound

r_j for j odd forms a decreasing sequence, i.e.

$$r_1 > r_3 > r_5 > r_7 > ---$$

and $\lim_{N\to\infty} r_N$ exists and for every $j \geq 0$

$$r_{2j} < \lim_{N\to\infty} r_N < r_{2j+1} \qquad \text{(j an integer)}$$

i.e.

$$r_o < r_2 < r_4 < --- < \lim_{N\to\infty} r_N < --- < r_5 < r_3 < r_1$$

In order to prove theorem 5, the following two lemmas must be shown:

Lemma (1): The following equations hold for $i \geq 1$

$$h_iK_{i-1} - h_{i-1} K_i = (-1)^{i-1} \qquad (10a)$$

which implies

$$r_i - r_{i-1} = \frac{(-1)^{i-1}}{K_i K_{i-1}} \qquad (10b)$$

Lemma (2): for $i \geq 2$:

$$h_iK_{i-2} - h_{i-2} K_i = (-1)^i a_i \qquad (11a)$$

which implies:

$$r_i - r_{i-2} = \frac{(-1)^i a_i}{K_i K_{i-2}} \qquad (11b)$$

First to prove (10a-b). Using equations (8a-b), the proof will follow by mathematical induction:
For i=-1, it is easily shown that:

$$h_{-1} K_{-2} - h_{-2} K_{-1} = 1$$

Also using equations (8a-b), the following result holds for i=0:

$$h_0 K_{-1} - h_{-1} K_o = -1$$

Now assume:

$$h_{i-1} K_{i-2} - h_{i-2} K_{i-1} = (-1)^{i-2}$$

using equations (8a-b) implies:

$$h_iK_{i-1} - h_{i-1} K_i = (a_ih_{i-1}+h_{i-2})K_{i-1} - (h_{i-1})(a_iK_{i-1} + K_{i-2})$$

$$= a_ih_{i-1}K_{i-1} + h_{i-2} K_{i-1} - a_iK_{i-1}h_{i-1} - h_{i-1} K_{i-2}$$

$$= -(h_{i-1}K_{i-2} - h_{i-2} K_{i-1}) = (-1)^{i-1}$$

since $h_iK_{i-1} - h_{i-1} K_i = (-1)^{i-1}$

this implies:

$$\frac{h_i}{K_i} - \frac{h_{i-1}}{K_{i-1}} = \frac{(-1)^{i-1}}{K_i K_{i-1}} = r_i - r_{i-1} \qquad (12)$$

Hence lemma 1 is shown. To prove lemma (2), it is observed that:

$$h_o K_{-2} - h_{-2} K_o = a_o$$

and equation (11a) holds for the case i=1 (which is easily verified); proceeding as before:.

$$h_i K_{i-2} - h_{i-2} K_i = (a_i h_{i-1} + h_{i-2})K_{i-2}$$
$$-h_{i-2}(a_i K_{i-1} + K_{i-2})$$
$$= a_i h_{i-1} K_{i-2} + h_{i-2} K_{i-2} - a_i h_{i-2} K_{i-1} - h_{i-2} K_{i-2}$$
$$= a_i [h_{i-1}K_{i-2} - h_{i-2} K_{i-1}] = a_i [(-1)^i]$$

which implies·

$$h_i K_{i-2} - h_{i-2} K_i = (-1)^i a_i$$

or

$$\frac{h_i}{K_i} - \frac{h_{i-2}}{K_{i-2}} = \frac{(-1)^i a_i}{K_i K_{i-2}}$$

or

$$r_i - r_{i-2} = \frac{(-1)^i a_i}{K_i K_{i-2}} \tag{13}$$

Hence the results of lemma 2 hold. Now the proof of theorem 5 will be demonstrated:
Since each $K_i > 0$, $a_i > 0$

we know

$$r_i - r_{i-1} = \frac{(-1)^{i-1}}{K_i K_{i-1}} \quad \text{and} \quad r_i - r_{i-2} = \frac{(-1)^i a_i}{K_i K_{i-2}}$$

It follows for j= an integer:

$$r_{2j} < r_{2j+2} \quad \text{from equation (13).}$$

From equations (12,13), we know for $a_i > 0$ and j= an integer:

$$r_{2j-1} > r_{2j+1} \quad \text{and} \quad r_{2j} < r_{2j-1}$$

Therefore:

$$r_o < r_2 < r_4 < \text{---} \quad \text{and} \quad r_1 > r_3 > r_5 > \text{---}$$

It remains to show that:

$$r_{2n} < r_{2n+2j} < r_{2n+2j-1} \leq r_{2j-1}$$

The sequence r_o, r_2, r_4,--- is monotonically increasing and is bounded above by r_1 and since all $r_i > 0$, it has a limit [4]. In a similar manner the sequence r_1, r_3, r_5,--- is monotonically decreasing and is bounded below by r_o and thus has a limit. The two limits are equal because:

$$\lim_{i \to \infty} [r_i - r_{i-1}] = \frac{(-1)^{i-1}}{K_i K_{i-1}} \to 0$$

as $i \to \infty$

Since each $K_i > K_{i-1}$; the proof of theorem 3 is complete.

Before examples are worked it is necessary to consider the periodicity of the riccati equation in the coefficients of the continued fraction expansion.

Theorem 6

The continued fraction which represents a quadratic surd is periodic. The term quadratic surd from number theory [3] means a quadratic equation similar to the riccati type.

Proof:

If $x = \langle a_o, a_1, a_2, \text{---}, a_n, a_o, a_1, \text{---}, a_n, a_o, \text{---} \rangle$

Let $a_L = \langle a_o, a_1, a_2, \text{---}, a_n \rangle$

Then $x = \langle a_L, x \rangle$ implies

$$x = \frac{x h_n + h_{n-1}}{x K_n + K_{n-1}}$$

or x satisfies an equation of the form:

$$\bar{A} x^2 + \bar{B} x + c = 0 \quad \text{where } \bar{A} \neq 0$$

because each $K_i > 0$. The converse of this theorem can also be shown but the proof is more difficult.

5. A Scalar Example

Example 1:

Consider the solution to the following riccati equation by the continued fraction approach:

$$a = -1, \quad r=2, \quad q=1 \quad \text{i.e.}$$
$$-p - p - 2p^2 + 1 = 0$$

or $2p^2 + 2p - 1 = 0$ which has roots

$$p = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}$$

The positive root can be written p=.366---
Now using the continued fraction approach, we know p satisfies the following equation:

$$p = \frac{w}{2} - \frac{1}{2}$$

and w satisfies:

$$w^2 = qr + a^2$$
$$=(1)(2)+1 \tag{14}$$
$$w^2 = 3$$

To illustrate the applicability of this approach, if

$$w^2 = qr + a^2$$

or

$$w = \sqrt{3}$$
$$w = 1 + (\sqrt{3} - 1) \tag{15}$$

or

$$w = 1 + \cfrac{1}{\frac{1}{2}\left[\sqrt{3}+1\right]} = 1 + \cfrac{1}{\frac{1}{2} + \frac{1}{2}\left[1 + \sqrt{3} - 1\right]}$$

$$= 1 + \cfrac{1}{1 + \frac{1}{2}\left[\sqrt{3}-1\right]} = 1 + \cfrac{1}{1 + \frac{1}{2}\left[\cfrac{1}{\frac{1}{2}\left[\sqrt{3}+1\right]}\right]}$$

$$= 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \text{---}}}}}$$

Since $\sqrt{3} = 1.7320508$----

Then the continued fraction of interest is:

$$1.732\text{---} \cong 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \text{---}}}}} \tag{16}$$

Now it is necessary to calculate the K_n and h_n terms. Table I illustrates this calculation. Now the concept of "best possible approximations" will be considered for the scalar riccati equation and for any irrational number.

Example 2

Let ξ_o be the irrational number π, i.e.

$$\xi_o = 3.14+\text{---}$$

Hence $a_o = [\xi_o] = 3$. To calculate a_1, compute

$$\xi_1 = \frac{1}{\pi - a_o}$$

$$\xi_1 = \frac{1}{\pi - 3.} = \frac{1}{.14159} = 7.06+\text{---}$$

TABLE I

h_i	K_i				a_i
$h_{-2} = 0$	$K_{-2} = 1$	r_n	$r_i - r_{i-1}$	$r_i - r_{i-2}$	
$h_{-1} = 1$	$K_{-1} = 0$				
$h_0 = 1$	$K_0 = 1$	1.0			$a_0 = 1$
$h_1 = 2$	$K_1 = 1$	2.0	1.0		$a_1 = 1$
$h_2 = 5$	$K_2 = 3$	1.667	-.333	.6667	$a_2 = 2$
$h_3 = 7$	$K_3 = 4$	1.750	.08333	-.2500	$a_3 = 1$
$h_4 = 19$	$K_4 = 11$	1.7272	-.0227	.06060606	$a_4 = 2$
$h_5 = 26$	$K_5 = 15$	1.7333	.0060606	-.0166666	$a_5 = 1$
$h_6 = 71$	$K_6 = 41$	1.7317	-.001626	.00434589	$a_6 = 2$
$h_7 = 97$	$K_7 = 56$	1.73214	.000439	-.0011905	$a_7 = 1$
$h_8 = 265$	$K_8 = 153$	1.732026	-.0001167	.0003188267	$a_8 = 2$
$h_9 = 362$	$K_9 = 209$	1.732057	.00003127	-.0000854409	$a_9 = 1$
$h_{10} = 989$	$K_{10} = 571$	1.732049	-.0000083795	.000022893	$a_{10} = 2$

Using

$$r_n = \frac{h_n}{K_n} \quad \text{for } n \geq o$$

$$r_i - r_{i-1} = \frac{(-1)^{i-1}}{K_i\, K_{i-1}}$$

$$r_i - r_{i-2} = \frac{(-1)^i\, a_i}{K_i\, K_{i-2}}$$

6. The "Best Possible Approximation"

Given an irrational number ξ, the following continued fraction expansion is "best possible" [5] for numerical computation of ξ:

Procedure:

Using the notation $a_i = [\xi_i]$ where a_i denotes the nearest integer smaller than the irrational number ξ_i, then a_i (i=0,---) is computed as follows:

$$a_o = [\xi_o], \text{ now proceeding as an algorithm,}$$

let

$$\xi_1 = \frac{1}{\xi_o - a_o} \quad \text{and} \quad a_1 = [\xi_1]$$

inductively it follows that

$$\xi_{i+1} = \frac{1}{\xi_i - a_i} \quad \text{and} \quad a_i = [\xi_i]$$

An example will now be worked with the irrational number π to illustrate this approach.

Now compute $a_1 = [\xi_1] = 7$. In this manner we calculate (using double precision (29 digits) on a CDC 6600 computer) the following numerical results:

$$\begin{aligned}
\pi = \xi_o &= 3.1415926535897932384626433832 79 \\
\xi_1 &= 7.0625133059310457697930051531 \\
\xi_2 &= 15.996594406685694113105998 74 \\
\xi_3 &= 1.0034172310133977856369454934 \\
\xi_4 &= 292.63459101223866070785852178 \\
\xi_5 &= 1.5758180949841629954527421461 \\
\xi_6 &= 1.7366595609113341887024369765 \\
\xi_7 &= 1.3574791573503535151332712575 \\
\xi_8 &= 2.7973658867611542733590927681 \\
\xi_9 &= 1.2541293985649846777332901666 \\
\xi_{10} &= 3.9350032135077287460064498245 \\
\xi_{11} &= 1.0695150407541735890218572566 \\
\xi_{12} &= 14.385376015764735321479897720
\end{aligned}$$

This gives rise to $a_o=3$, $a_1=7$, $a_2=15$, $a_3=1$, $a_4=292$, $a_5=1$, $a_6=1$, $a_7=1$, $a_8=2$, $a_9=1$, $a_{10}=3$, $a_{11}=1$, $a_{12}=14$

which results in the following partial fraction expansion:

$$\pi = 3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{14+ \cdots}}}}}}}}}}}}$$

The first 8 terms are calculated and expanded (the computations are excessive).

$$\begin{aligned}
r_o &= 3 = 3.0 \\
r_1 &= 22/7 = 3.142857142857\text{---} \\
r_2 &= 333/106 = 3.14150943396226415094\text{---} \\
r_3 &= 355/113 = 3.14159292035\text{---} \\
r_4 &= 103993/33102 = 3.14159265301l\text{---} \\
r_5 &= 104348/33215 = 3.14159265392142\text{---} \\
r_6 &= 208341/66317 = 3.141592653467\text{----} \\
r_7 &= 312689/99532 = 3.1415926536189\text{---} \\
r_8 &= 833719/265381 = 3.14159265358107777\text{--}
\end{aligned}$$

It is noted that 9 terms of the fraction expansion give rise to accuracy beyond 9 digits. The first 4 terms gave an accuracy of 7 decimal places. It can be explicitly shown [5] that this numerical approach is the most efficient method to numerically calculate an irrational number to any number of decimal places. The proof will not be given here. Now this approach will be worked for the solution to the riccati equation (14).

Example 3

Expanding the square root of 3 to 30 decimal places yields:

$$\xi_o = \sqrt{3} = 1.732050807568877293527446341505$$

Now the calculation of the a_i and ξ_i will proceed as for π .

$$a_o = [1.732\text{---}] = 1$$

$$\xi_1 = \frac{1}{\sqrt{3}-1} = 1.36\text{+---}$$

Hence

$$a_1 = [\xi_1] = 1.$$

A summary of the first 11 terms yields:

$$\begin{aligned}
\xi_1 &= 1.366025403784436347975702660 5 \\
\xi_2 &= 2.732050807568899213767313378 0 \\
\xi_3 &= 1.366025403784396571251840576 6 \\
\xi_4 &= 2.732050807569190909742302038 8 \\
\xi_5 &= 1.366025403783852956025725661 7 \\
\xi_6 &= 2.732050807573248135576241687 7 \\
\xi_7 &= 1.366025403776282119584023896 3 \\
\xi_8 &= 2.732050807629757601277660404 2 \\
\xi_9 &= 1.366025403670834024635035129 4 \\
\xi_{10} &= 2.732050808416832895506520796 9 \\
\xi_{11} &= 1.366025402202131533482739200 8
\end{aligned}$$

Which results in $a_o=1$, $a_1=1$, $a_2=2$, $a_3=1$, $a_4=2$, $a_5=1$, $a_6=2$, $a_7=1$, $a_8=2$, $a_9=1$, $a_{10}=2$.

In this manner, the "most efficient" method to calculate the solution to the riccati equation (14) is as follows using the continued fraction approach:

$$\xi = 1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+ \cdots}}}}$$

It is noted that this expansion is identical to the one given in equation (16). This result occurs due to the following identity:

$$1+\sqrt{3} = \cfrac{1}{\cfrac{1}{\sqrt{3}-1} - 1}$$

which results in: $\xi_j = \xi_{j+2}$ for all j

with $a_i = 1$ (i odd)
$a_{i+1} = 2$ (i odd)

Since the square root of 3 is an irrational number (as the solution of any quadratic equation may be), the "most efficient" method to numerically compute the solution to this riccati equation has been demonstrated by equation (16). This technique is numerically illustrated here in an effort to develop an algorithm for the matrix case and when the solution to equation (2) is not known apriori.

7. Conclusions

A continued fraction approach is used to investigate the solution to the steady state riccati equation. Theorems are proven to demonstrate an algorithm which converges for the steady state scalar case. Some examples are worked to demonstrate the "most efficient" method to compute the solution to the riccati equation.

References

[1] D. G. Lainiotis, "Discrete Riccati Equation Solutions:Partitioned Algorithms", IEEE Transactions on Automatic Control, August, 1975, pp. 555-556

[2] D. G. Lainiotis, "Partitioned Estimation Algorithms, I: Nonlinear Estimation," Journal of Information Sciences, Vol. 7, No. 3, 1974.

[3] G. H. Hardy and E. M. Wright, "An Introduction to The Theory of Numbers", Oxford University Press, 1960.

[4] L. V. Kantorovich and G. P. Akilov, "Functional Analysis in Normed Spaces", New York: MacMillan, 1964, pp. 189-190.

[5] I. Niven and H. S. Zuckerman, "An Introduction To The Theory of Numbers", John Wiley & Sons, 1966.